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Who matters in coordination problems?

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Abstract: Agents face a coordination problem akin to the adoption of a network technology. A principal announces investment subsidies that, at minimal cost, attain a given likelihood of successful coordination. Optimal subsidies target agents who impose high externalities on others and on whom others impose low externalities. Based on the analysis of the role of strategic uncertainty in coordination processes, we provide a methodology that can be used to find the optimal targets for a variety of interventions in a large class of coordination problems with heterogeneous agents.

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Who Matters in Coordination Problems?
Web Appendix

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Appendices

Proof of Proposition 1

We prove our characterization result for a broader class of payoffs than those in the baseline specification of Section II. We allow the payoffs for investing to be of the following form

$$u_g(a,\theta) = \begin{cases} \overline{u}_g(a,\theta) & \text{if } a \geq 1-\theta, \\ \underline{u}_g(a,\theta) & \text{if } a < 1-\theta. \end{cases} \tag{A1}$$

Payoff for not investing is 0 as before. The information structure is unaltered.

We will divide the proof into three parts. Lemma 6 proves equilibrium existence and uniqueness, Lemma 7 establishes convergence of equilibrium as $\sigma \to 0$, and Proposition 3 provides the characterization of the limit equilibrium. The lemmas are proved under an increasing set of assumptions specified below.

A1. Preliminaries

We rescale the function of the aggregate action as follows: $\tilde{a}(\zeta,\Delta) = \sum_g w_g m_g \left(1 - F_g\left(\Delta_g - \zeta\right)\right)$; when $x_g^* = x_1^* + \sigma\Delta_g$ and $\theta = x_1^* + \sigma\zeta$ then the aggregate action $\hat{a}(\theta,\mathbf{x}^*) = \tilde{a}(\zeta,\Delta)$. For $a \in (0,1)$, we let $\vartheta(a,\Delta)$ be the inverse function to $\tilde{a}(\zeta,\Delta)$ with respect to ζ. It is well defined and increasing in a because $\tilde{a}(\zeta,\Delta)$ is increasing in ζ when $\tilde{a}(\zeta,\Delta) \in (0,1)$.

The strategic beliefs can be written as

$$A_g(a,\Delta) = \Pr\left(\tilde{a}\left(\Delta_g - \eta_g,\Delta\right) < a\right), \text{ where } \eta_g \sim F_g. \tag{A2}$$

Lemma 4. *For any Δ, the densities associated with the strategic beliefs are bounded: $0 \leq \frac{\partial}{\partial a}A_g(a,\Delta) \leq 1/(w_g m_g)$.*

Proof of Lemma 4. Using (A2), we can write $A_g(a,\Delta) = 1 - F_g(\Delta_g - \vartheta(a,\Delta))$. Differentiating gives

$$\frac{\partial}{\partial a}A_g(a,\Delta) = f_g(\Delta_g - \vartheta(a,\Delta))\frac{\partial}{\partial a}\vartheta(a,\Delta) = \frac{f_g(\Delta_g - \vartheta(a,\Delta))}{\sum_{g'} w_{g'} m_{g'} f_{g'}(\Delta_{g'} - \vartheta(a,\Delta))}.$$

The last expression lies in $[0, 1/w_g m_g]$ because $f_{g'}(\cdot)$ is non-negative for all g' and positive for some g' when $0 < a < 1$.

Finally, we introduce notation for the payoff expectation of the threshold type from group g:

$$H_g^\sigma(x_1, \Delta) = E\left[u_g(a,\theta)\,\middle|\,(x_g^*, g)\right] = \int_0^1 u_g(a, x_1 + \sigma\vartheta(a,\Delta))dA_g(a,\Delta).$$

Functions $H_g^\sigma(x_1, \Delta)$ are well defined even for $\sigma = 0$ because the beliefs $A_g(a,\Delta)$ are independent of σ.

A2. Equilibrium Existence, Uniqueness, and Monotonicity

We prove equilibrium existence, uniqueness, and monotonicity under the following assumptions:
Assumption 0: $\overline{u}_g(a,\theta)$ is positive and $\underline{u}_g(a,\theta)$ is negative.
Assumption 1: Both $\overline{u}_g(a,\theta)$, and $\underline{u}_g(a,\theta)$ are non-decreasing in both arguments.
Assumption 2: The difference $\overline{u}_g(a,\theta) - \underline{u}_g(a,\theta)$ is bounded by positive constants; $0 < \underline{b} < \overline{u}_g(a,\theta) - \underline{u}_g(a,\theta) < \overline{b}$.
Assumption 3: $-\underline{u}_g(a,\theta)$ is bounded by positive constants; $0 < \underline{c} < -\underline{u}_g(a,\theta) < \overline{c} < \underline{b}$.
Assumption 1 implies that the incentive to invest, $u_g(a,\theta)$, is non-decreasing both in the aggregate action and in the fundamental. Assumptions 2 and 3 imply that each player strictly prefers to invest whenever she assigns probability exceeding $\overline{p} = \overline{c}/\underline{b} < 1$ to success (that is, to $a \geq 1-\theta$) and she prefers not to invest whenever she assigns probability less than $\underline{p} = \underline{c}/\overline{b}$ to success. Also, Assumptions 2 and 3 imply that the functions u_g are bounded. Notice that Assumptions 0-3 are satisfied in the baseline setup.

We first establish that the system of indifference conditions does not have multiple solutions. This holds even when $\sigma = 0$, which will be useful in the analysis of the limit of the equilibria, as $\sigma \to 0$.

Lemma 5. *If Assumptions 0-3 hold then for each $\sigma \in [0,1]$, the system of indifference conditions,*

$$H_g^\sigma(x_1,\Delta) = 0 \textit{ for all } g,$$

has at most one solution $x_1^*(\sigma)$, $\Delta^*(\sigma)$.

The proof adapts the translation argument from Frankel, Morris, and Pauzner (2003).

Proof of Lemma 5. Assume the existence of two distinct solutions (x_1, Δ) and (x_1', Δ'). Using a translation of one of the solutions, we demonstrate a contradiction. We distinguish two cases, $\Delta = \Delta'$, and $\Delta \neq \Delta'$.

If $\Delta = \Delta'$ then $x_1 \neq x_1'$ and, without loss of generality, $x_1 < x_1'$. Recalling $\theta = x_1 + \sigma\vartheta(a,\Delta)$, we define $a^*, a'^* \in (0,1)$ as the unique solutions of $a = 1 - x_1 - \sigma\vartheta(a,\Delta)$, and $a = 1 - x_1' - \sigma\vartheta(a,\Delta)$, respectively.[1] The project succeeds whenever $a > a^*$, or

[1] The solutions exist; otherwise the threshold type's payoff expectation would be strictly positive or negative, violating the indifference condition. The solution is unique because $a + \sigma\vartheta(a,\Delta)$ is strictly monotone in a.

$a > a'^*$ under the first and the second profile, respectively. Notice that $a^* > a'^*$, because $a + \sigma\vartheta(a, \Delta)$ strictly increases in a and $x_1 < x_1'$.

Both profiles satisfy the indifference conditions, so $H_g^\sigma(x_1', \Delta) - H_g^\sigma(x_1, \Delta) = 0$. But, for any a, the incentive to invest, $u_g(a, x_1' + \sigma\vartheta(a, \Delta))$ under the profile (x_1', Δ) is at least as large as the incentive $u_g(a, x_1 + \sigma\vartheta(a, \Delta))$ under the profile (x_1, Δ), because $x_1' > x_1$ and u_g is non-decreasing. Furthermore, by Assumption 2, $u_g(a, x_1' + \sigma\vartheta(a, \Delta)) - u_g(a, x_1 + \sigma\vartheta(a, \Delta)) \geq \underline{b}$ for $a \in (a'^*, a^*)$. Thus

$$H_g^\sigma(x_1', \Delta) - H_g^\sigma(x_1, \Delta) \geq \underline{b}\left(A_g(a^*, \Delta) - A_g(a'^*, \Delta)\right).$$

This establishes a contradiction because $A_g(a^*, \Delta) - A_g(a'^*, \Delta) > 0$. The strict inequality holds because a^* is in the interior of the support of $A_g(a, \Delta)$, otherwise the critical type would assign probability 0 or 1 to the success.

In the second case, $\Delta \neq \Delta'$, and, without loss of generality, $x_1 \leq x_1'$. Choose $h \in \arg\max_g(\Delta_g' - \Delta_g)$, and let $D = \max_g(\Delta_g' - \Delta_g)$. Notice that $D \geq 0$ because $\Delta_1' - \Delta_1 = 0$ by definition. Observe that $\Delta_h' - \Delta_g' \geq \Delta_h - \Delta_g$ for all g, and the inequality is strict for at least one g because $\Delta \neq \Delta'$. Letting $\tilde{x}_1 = x_1' + \sigma D$, we have

$$H_h^\sigma(\tilde{x}_1, \Delta) \geq H_h^\sigma(x_1, \Delta) = 0,$$

because $\tilde{x}_1 \geq x_1' \geq x_1$ and H is non-decreasing in the first argument. We show for contradiction that

$$H_h^\sigma(\tilde{x}_1, \Delta) < H_h^\sigma(x_1', \Delta') = 0. \tag{A3}$$

Recall that $H_h^\sigma(\tilde{x}_1, \Delta) = \int_0^1 u_h(a, \tilde{x}_1 + \sigma\vartheta(a, \Delta)) dA_h(a, \Delta)$. The substitution $a = \tilde{a}(\Delta_h - \eta, \Delta)$ gives

$$H_h^\sigma(\tilde{x}_1, \Delta) = \int_{-1/2}^{1/2} u_h\left(\tilde{a}(\Delta_h - \eta, \Delta), \tilde{x}_h - \sigma\eta\right) dF_h(\eta), \tag{A4}$$

where $\tilde{x}_h = \tilde{x}_1 + \sigma\Delta_h$. Similarly,

$$H_h^\sigma(x_1', \Delta') = \int_{-1/2}^{1/2} u_h\left(\tilde{a}(\Delta_h' - \eta, \Delta'), x_h' - \sigma\eta\right) dF_h(\eta), \tag{A5}$$

where $x_h' = x_1' + \sigma\Delta_h'$.

To establish inequality (A3), we will use $\tilde{x}_h = x_h'$, and

$$\tilde{a}(\Delta_h' - \eta, \Delta') \geq \tilde{a}(\Delta_h - \eta, \Delta) \text{ for all } \eta. \tag{A6}$$

The equality $\tilde{x}_h = x_h'$ holds because $\tilde{x}_h = x_1' + \sigma D + \sigma\Delta_h = x_1' + \sigma(\Delta_h' - \Delta_h) + \sigma\Delta_h = x_h'$. Inequality (A6) holds because, using the definition of the function $\tilde{a}$, (A6) can be rewritten as

$$\sum_g w_g m_g \left(1 - F_g(\Delta'_g - \Delta'_h + \eta)\right) \geq \sum_g w_g m_g \left(1 - F_g(\Delta_g - \Delta_h + \eta)\right).$$

This in turn holds because $\Delta'_h - \Delta'_g \geq \Delta_h - \Delta_g$ for all g.

Define η^* as the unique solution of $\tilde{a}(\Delta_h - \eta, \Delta) = 1 - (\tilde{x}_h - \sigma\eta)$, and η'^* as the unique solution of $\tilde{a}(\Delta'_h - \eta, \Delta') = 1 - (x'_h - \sigma\eta)$. The equality $\tilde{x}_h = x'_h$, and the inequality (A6) implies $\eta^* \leq \eta'^*$.

We can further strengthen the last inequality to the strict one, $\eta^* < \eta'^*$. Assume for contradiction $\eta^* = \eta'^*$. Recall that there exists g such that $\Delta'_h - \Delta'_g > \Delta_h - \Delta_g$ because $\Delta' \neq \Delta$. Then

$$w_g m_g \left(1 - F_g(\Delta'_g - \Delta'_h + \eta^*)\right) > w_g m_g \left(1 - F_g(\Delta_g - \Delta_h + \eta^*)\right)$$

because $\Delta'_g - \Delta'_h + \eta^*$ and $\Delta_g - \Delta_h + \eta^*$ must lie in the support of F_g, otherwise the critical type of group g would assign probability 0 or 1 to success. This implies that inequality (A6) is strict at η^*, contradicting the assumption $\eta^* = \eta'^*$.

Using (A4) and (A5), the equality $\tilde{x}_h = x'_h$, and the inequality (A6), we get

$$H_h^\sigma(x'_1, \Delta') - H_h^\sigma(\tilde{x}_1, \Delta) \geq \underline{b}\left(F_h\left(\eta'^*\right) - F_h\left(\eta^*\right)\right) > 0.$$

Lemma 5 implies equilibrium uniqueness within the class of monotone strategy profiles. The next lemma establishes that non-monotone equilibria do not exist. The proof is based on a standard "infection" argument.

Lemma 6. *If Assumptions 0-3 hold then the studied coordination game has a unique Bayes-Nash equilibrium, in which each player follows a threshold strategy:* $a_i(x,g) = \begin{cases} 1 & \text{if } x \geq x_g^*, \\ 0 & \text{if } x < x_g^*. \end{cases}$

Proof of Lemma 6. By Assumptions 2 and 3, a player strictly prefers to invest whenever she assigns probability higher than $\overline{p}$ to the success of the project. Hence investing is dominant for all types (x,g) with signal $x > \overline{x}_g^0 = 1 + \sigma F_g^{-1}(\overline{p})$.

Let $U_g(x, \mathbf{x}^*) = E\left[u_g(\hat{a}(\theta, \mathbf{x}^*), \theta)|\,(x,g)\right]$ be the expected payoff for investing of type (x,g) against the monotone strategy profile with thresholds $\mathbf{x}^*$. $U_g(x, \mathbf{x}^*)$ is non-decreasing in x, non-increasing in $\mathbf{x}^*$, and continuous. For $k \geq 0$, define $\overline{x}_g^{k+1}$ to be the unique solution of $U_g(x, \overline{\mathbf{x}}^k) = 0$. The sequence is well defined: the solution exists and is unique because $\lim_{x\to+\infty} U_g(x, \overline{\mathbf{x}}^k)$ is positive, $\lim_{x\to-\infty} U_g(x, \overline{\mathbf{x}}^k)$ is negative, U_g is continuous and strictly increasing in x at the indifference point. Moreover, the sequence $\overline{x}_g^k$ is bounded from below by $0 + \sigma F_g^{-1}\left(\underline{p}\right)$.

By induction, the sequence $\overline{x}_g^k$ is non-increasing: Notice that $\overline{x}_g^1 \leq \overline{x}_g^0$. Suppose $\overline{x}_g^k \leq \overline{x}_g^{k-1}$ for all g. Then $\hat{a}(\theta, \overline{\mathbf{x}}^k) \geq \hat{a}(\theta, \overline{\mathbf{x}}^{k-1})$. Hence, $U_g\left(x, \overline{\mathbf{x}}^k\right) \geq U_g\left(x, \overline{\mathbf{x}}^{k-1}\right)$ and the root of $U_g\left(x, \overline{\mathbf{x}}^k\right)$ does not exceed the root of $U_g\left(x, \overline{\mathbf{x}}^{k-1}\right)$.

Also by induction, investing is serially dominant for all types (x, g) with $x > \overline{x}_g^k$, for any k. Assume the statement holds for k. Then $\hat{a}\left(\theta, \overline{\mathbf{x}}^k\right)$ is a lower bound on the aggregate action under any profile that survives k iterations of deletion of dominated strategies and hence, investing is serially dominant for all (x, g) with x exceeding the root of $U_g\left(x, \overline{\mathbf{x}}^k\right)$.

The non-increasing, bounded sequence $\overline{x}_g^k$ has a limit denoted by $\overline{x}_g$. For each k, $U_g\left(\overline{x}_g^{k+1}, \overline{\mathbf{x}}^k\right) = 0$ by the definition of the sequence. Function U_g is continuous and hence the limit thresholds satisfy the indifference conditions: $U_g\left(\overline{x}_g, \overline{\mathbf{x}}\right) = 0$ for all g.

By the symmetric argument, there exists a vector of thresholds $\underline{\mathbf{x}}$ such that investing is serially dominated for all types (x, g) with $x < \underline{x}_g$, and $\underline{\mathbf{x}}$ satisfies the indifference conditions, $U_g\left(\underline{x}_g, \underline{\mathbf{x}}\right) = 0$. Lemma 5 establishes that there exists a unique $\mathbf{x}^*$ satisfying the indifference conditions. Hence $\underline{\mathbf{x}} = \overline{\mathbf{x}} = \mathbf{x}^*$.

When $\mathbf{x}^*$ satisfies the indifference conditions then the threshold strategy profile defined by $\mathbf{x}^*$ constitutes a Bayes-Nash equilibrium, because the incentive to invest, $U_g\left(x, \mathbf{x}^*\right)$ is non-decreasing in x. No other equilibria exist because investing is serially dominant above, and serially dominated below the thresholds.

A3. Equilibrium Convergence

We now establish equilibrium convergence as $\sigma \to 0$ under Assumptions 0-3 and one additional assumption:

Assumption 4: The functions $\overline{u}_g(a, \theta)$ and $\underline{u}_g(a, \theta)$ are Lipschitz continuous in θ.

Again, Assumptions 0-4 are satisfied in the baseline setup with step-like payoffs.

Recall that $\Delta_1 = 0$ by definition so that Δ can be identified with a $(G-1)$-dimensional vector.

Lemma 7. *As $\sigma \to 0$, the thresholds, $x_g^*(\sigma)$, of all groups converge to a common critical fundamental, θ^*, while the relative positions of thresholds, $\left(x_g^*(\sigma) - x_1^*(\sigma)\right)/\sigma$, converge to some Δ_g^*, for each group. The G variables θ^* and $\Delta_2^*, \ldots, \Delta_G^*$ are the unique solution of the system of G limit indifference conditions*

$$\int_0^1 u_g(a, \theta^*) dA_g\left(a, \Delta^*\right) = 0, \text{ for all } g,$$

where the strategic beliefs $A_g\left(a, \Delta^*\right)$ are defined by (7) *in the main text.*

Proof of Lemma 7. We first establish that $(x_1^*(\sigma), \Delta(\sigma))$ lies in a compact set S, uniformly across all $\sigma \in (0, 1]$. All critical types x_g^* lie in $\sigma/2$-neighborhood of the critical fundamental $\theta^*(\sigma)$; otherwise the critical type would know the outcome of the project and thus violate the indifference condition. Thus $\Delta_g \in [-1, 1]$ for all g. Finally, $x_1^*(\sigma)$ is bounded as well; for all $\sigma \in (0, 1]$, $x_1^*(\sigma) \in \left[\sigma F_0^{-1}(\underline{p}), 1 + \sigma F_0^{-1}(\overline{p})\right] \subset [-1/2, 3/2]$.

Next, we establish that, when σ is small, the indifference conditions, $H_g^\sigma(x_1, \Delta) = 0$, are well approximated by the limit conditions, $H_g^0(x_1, \Delta) = 0$. We prove that $H_g^\sigma(x_1, \Delta)$

converges uniformly to $H_g^0(x_1,\Delta)$ on $\mathbf{R}\times[-1,1]^{G-1}$:

$$|H_g^\sigma(x_1,\Delta)-H_g^0(x_1,\Delta)|\le\int_0^1|u_g(a,x_1+\sigma\vartheta(a,\Delta))-u_g(a,x_1)|\,dA_g(a,\Delta).$$

We decompose the difference $u_g(a,x_1+\sigma\vartheta(a,\Delta))-u_g(a,x_1)$ into two parts:

$$|\overline{u}_g(a,x_1+\sigma\vartheta(a,\Delta))-\overline{u}_g(a,x_1)|\le c_1\sigma,$$

where c_1 is a positive constant, because $\overline{u}_g$ is Lipschitz continuous and ϑ is bounded (as all thresholds lie in the $\sigma/2$-neighborhood of $\theta^*(\sigma)$); similarly for $\underline{u}_g$. Additionally, $u_g(a,x_1+\sigma\vartheta(a,\Delta))$ and $u_g(a,x_1)$ may differ because the events of success $a\ge 1-(x_1+\sigma\vartheta(a,\Delta))$, and $a\ge 1-x_1$, respectively, arise for different sets of a in the two cases. Denote by $R(a,\theta)$ the outcome function; $R(a,\theta)=1$ if $a\ge 1-\theta$, and 0 otherwise.

$$\int_0^1|R(a,x_1+\sigma\vartheta(a,\Delta))-R(a,x_1)|\,dA_g(a,\Delta)\le c_2\sigma$$

for some positive c_2 because the derivative of $A_g(a)$ is bounded by $1/(w_g m_g)$ from above and ϑ is bounded. Altogether $|H_g^\sigma(x_1,\Delta)-H_g^0(x_1,\Delta)|\le(c_1+c_2\overline{b})\sigma$.

Let θ^* and Δ^* denote the solution of the indifference conditions $H_g^0(x_1,\Delta)=0$. Given any neighborhood N of (θ^*,Δ^*), function $H_g^0(x_1,\Delta)$ is uniformly bounded from 0 by some ϵ on $S\setminus N$. Choosing $\overline{\sigma}$ such that $\left|H_g^\sigma(x_1,\Delta)-H_g^0(x_1,\Delta)\right|<\epsilon$ on S for all $\sigma<\overline{\sigma}$, the system of equations $H_g^\sigma(x_1,\Delta)=0$ has no solution outside of N.

A4. Equilibrium Characterization

Finally, we characterize the critical state θ^* in the limit, as $\sigma\to 0$. We impose an additional assumption that restricts the extent of heterogeneity of investment incentives. It states that the incentive to invest varies with a homogeneously across the groups, up to a scale factor $\beta(\theta)$.

Assumption 5: There exists a function $u(a,\theta)$ and positive functions $\beta_g(\theta)$, $\gamma_g(\theta)$ such that $u_g(a,\theta)=\beta_g(\theta)u(a,\theta)-\gamma_g(\theta)$ for all groups g.

Proposition 3. *If Assumptions 0-5 hold then, for each σ, the coordination game has a unique Bayes-Nash equilibrium, in which players use threshold strategies. As $\sigma\to 0$ all thresholds x_g^* converge to a common limit θ^*, which is the unique solution of*

$$\int_0^1 u(a,\theta^*)da=\sum_g w_g m_g\frac{\gamma_g(\theta^*)}{\beta_g(\theta^*)}. \tag{A7}$$

Proposition 1 in the main text is a special case of Proposition 3.

Proof of Proposition 3. Lemmas 6 and 7 state that, as $\sigma \to 0$, thresholds $x_g^*(\sigma)$ converge to θ^* that solves

$$\beta_g(\theta^*) \int_0^1 u(a, \theta^*) dA_g(a, \Delta^*) - \gamma_g(\theta^*) = 0,$$

for all g. Rearranging, multiplying by $w_g m_g$, and summing through g gives

$$\int_0^1 u(a, \theta^*) d \left(\sum_g w_g m_g A_g(a, \Delta^*) \right) = \sum_g w_g m_g \frac{\gamma_g(\theta^*)}{\beta_g(\theta^*)}.$$

This further simplifies into (A7) because $\sum_g w_g m_g A_g(a, \Delta^*) = a$ by the belief constraint.

Optimal Subsidy Scheme with General Payoffs

In Section IV of the main text, we solved the planner's problem for the simple step-like payoff functions of the baseline model. Here, we extend the solution to the general payoffs specified in (A1). For simplicity, we restrict attention to the case of only two groups.

As in Section IV, the planner specifies subsidies $s_g \geq 0$, which change the payoff for investing from $u_g(a, \theta)$ to $u_g(a, \theta) + s_g$. We focus on limited subsidies that do not make investing dominant: recall that $\underline{u}_g$ is bounded from above by $-\underline{c}$; the planner chooses subsidies $s_g \in [0, \underline{c})$. The planner chooses θ^* so that there exists a feasible scheme $\mathbf{s}$ implementing θ^*, and she solves the expenditure minimization problem (11) in the main text.

Proposition 4. *Assume that the population of players consists of two groups, and that A0-4 hold. Suppose $u_1(a, \theta)/w_1$ is less steep than $u_2(a, \theta)/w_2$ with respect to a. That is, $u_2(a, \theta)/w_2 - u_1(a, \theta)/w_1$ is non-decreasing in a, and strictly increasing for some range of a. Then the optimal policy solving problem* (11) *exclusively subsidizes group 1.*

The proposition does not rely on the relatively restrictive assumption A5 and thus holds even beyond the class of payoffs for which we derived the explicit equilibrium characterization.

Proof of Proposition 4. We consider scheme (s_1, s_2) with $s_2 > 0$ and find non-negative (s_1', s_2') with $s_2' < s_2$ implementing the same critical fundamental as (s_1, s_2), but with lower expenditures, $\sum_g m_g s_g' < \sum_g m_g s_g$.

The threshold types satisfy the indifference conditions under both schemes:

$$\int_0^1 u_g(a, \theta^*) dA_g(a, \Delta) + s_g = 0 = \int_0^1 u_g(a, \theta^*) dA_g(a, \Delta') + s_g'.$$

Multiplying the indifference conditions by m_g, summing through g, and rearranging, we relate the difference in subsidy expenditures to the difference in beliefs:

$$\sum_g m_g(s_g - s'_g) = \sum_g \int_0^1 \frac{u_g(a,\theta^*)}{w_g} m_g w_g d\left(A_g(a,\Delta') - A_g(a,\Delta)\right). \tag{B1}$$

The central step of the proof relies on the belief constraint. It implies that any (weighted) change of beliefs of group 2 equals the opposite (weighted) change of beliefs of group 1:

$$m_2 w_2 \left(A_2(a,\Delta') - A_2(a,\Delta)\right) = m_1 w_1 \left(A_1(a,\Delta) - A_1(a,\Delta')\right).$$

Combining this with (B1) gives

$$\sum_g m_g(s_g - s'_g) = m_2 w_2 \int_0^1 \Big(\frac{u_2(a,\theta^*)}{w_2} - \frac{u_1(a,\theta^*)}{w_1}\Big) d\left(A_2(a,\Delta') - A_2(a,\Delta)\right).$$

By assumption, $u_2(a,\theta^*)/w_2 - u_1(a,\theta^*)/w_1$ is non-decreasing in a, and strictly increasing for some range of a, and so to prove that $\sum_g m_g s_g > \sum_g m_g s'_g$ it suffices to show that $A_2(a,\Delta')$ first-order stochastically dominates $A_2(a,\Delta)$.

Notice from (7) that the strategic beliefs are monotone functions of Δ_2 and hence $A_2(a,\Delta)$ and $A_2(a,\Delta')$ are ordered by stochastic dominance. The subsidy $s'_2 < s_2$ and thus, to keep the threshold type $(x_2^*, 2)$ indifferent, her belief $A_2(a,\Delta')$ must indeed stochastically dominate $A_2(a,\Delta)$.

Critical State for Targeted Deposit Freeze in Section VI.A

Let a_g be proportion of group g that decides *not* to withdraw. Then the aggregate volume of the deposits kept in the bank (voluntarily or involuntarily) is $a = \sum_g m_g \left[1 - q_g + q_g a_g\right]$. Given the modified definition of the aggregate action a, the belief constraint in this situation is amended as follows. Let $A_g(a)$ be the belief (c.d.f.) over a of the critical type from group g. Then, for $a \in \left[\sum_g m_g(1-q_g), 1\right]$,

$$\sum_g m_g q_g A_g(a) = a - \sum_g m_g(1-q_g).$$

Recall that $p_g = 1 - A_g(1-\theta^*)$ is the success probability as evaluated by the critical type of group g. Thus

$$\sum_g m_g q_g p_g = \sum_g m_g q_g \left(1 - A_g(1-\theta^*)\right) = \sum_g m_g q_g - (1-\theta^*) + \sum_g m_g(1-q_g) = \theta^*.$$

As before, the indifference conditions imply that the success probabilities are $p_g = c_g/b_g$. Thus, the critical state is $\theta^* = \sum_g m_g q_g c_g / b_g$.

*

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